Jakub Löwit | **Tannakian Formalism** 5.3.2025

Literature: Deligne – Milne: Tannakian categories [some arguments "a bit sketchy"]
Szamuely: Galois groups and fundamental groups
Milne: Algebraic groups

Credit: attendance, talk in 6^{th} or 7^{th} week (to be scheduled)

0. Motivation

G algebraic group over a field k $\rightsquigarrow$ $\mathrm{Rep}_k(G)$ category of finite-dimensional algebraic representations of G

Questions:
- Can we reconstruct G from $\mathrm{Rep}_k(G)$?
- Given any category $\mathcal{C}$, can we see whether it is of the form $\mathrm{Rep}_k(G)$ for some G?
- Is this useful?

Answers: Yes,
- G is determined by $\mathrm{Rep}_k(G)$ regarded as a symmetric monoidal category under $\otimes$;
- if $\mathcal{C}$ has the structure of a neutral Tannakian category then it is of the form $\mathrm{Rep}_k(G)$;
- and this is useful in two ways:
  (1) neutral Tannakian categories arise in "nature", and we can study them in terms of representation theory;
  (2) and the formalism can give interesting models for $\mathrm{Rep}_k(G)$.

Example: Covering spaces

Let X be a connected, locally simply connected topological space.
Recall: A covering space of X is $(Y, Y \xrightarrow{f} X)$ such that for every $x \in X$ there exists an open neighbourhood U of x so that $f^{-1}(U) = \bigsqcup_i V_i$ and $f: V_i \xrightarrow{\sim} U_i$.

Want to understand: $\mathcal{C}$ category with $\begin{cases} \text{objects: } Y \xrightarrow{f} X \text{ covering spaces} \\ \text{morphisms: maps between them over } X. \end{cases}$

Fix a base point $x_0 \in X$ and consider $\pi_1 := \pi_1(X, x_0)$.

Then we have the following picture:

$$\begin{aligned} \omega_{x_0} : \{\text{covering spaces}\} &\xrightarrow{\simeq} \pi_1\text{-Sets} \\ (Y \xrightarrow{f} X) &\longmapsto \pi_1 \circlearrowright Y_{x_0} := f^{-1}(x_0). \end{aligned}$$

In fact, if we denote the universal covering space of X by $\tilde{X}$, then the inverse equivalence is given by $\tilde{X} \times S /_{\pi_1} \longleftarrow S$.

<u>Example</u>: <u>Separable field extensions</u>

Let k be a field. We now want to understand the category $\mathcal{C}$ with

$\begin{cases} \text{objects: finite separable field extensions } L/k \\ \text{morphisms: homomorphisms between them as } k\text{-algebras} \end{cases}$

⌜<u>Notation</u>: k^{sep} := separable closure, we <u>choose</u> $k \xrightarrow{i} k^{sep}$.

$\Gamma := \mathrm{Gal}(k^{sep}/k)$ absolute Galois group with profinite topology ⌟

For any L/k, consider the set of embeddings $\mathrm{Hom}_k(L, k^{sep})$, a finite set because L is assumed finite over k. Γ acts on this set by postcomposition, and the action is transitive (by separability) and continuous. We get the following picture:

$$\begin{aligned} \omega_i : \{L/k \text{ finite separable}\} &\xrightarrow{\simeq} \{\text{finite sets with continuous transitive } \Gamma\text{-action}\} \\ L/k &\longmapsto \Gamma \circlearrowright \mathrm{Hom}_k(L, k^{sep}) \end{aligned}$$

[where i was the chosen $k \hookrightarrow k^{sep}$], this is the main theorem of Galois theory.

For the inverse, note that any such Γ-set is of the form $\Gamma/\mathcal{U}$ for an open subgroup $\mathcal{U} \leq \Gamma$, and we map this to $(k^{sep})^{\mathcal{U}}$.

<u>1. Affine algebraic groups and their representations</u>

Throughout, k will be a field, $pt = \mathrm{Spec}\, k$.

<u>Def.</u> (geometric): An <u>affine algebraic group</u> G is an affine $\mathrm{Spec}\, A$ for a k-algebra A, together with maps

$$\begin{aligned} m &: G \times G \to G \\ e &: pt \to G \\ i &: G \to G \end{aligned}$$

satisfying the usual group axioms:

$$
\begin{tikzcd}
G\times G\times G \arrow[r, "m\times id"] \arrow[d, "id\times m"'] \arrow[dr, phantom, "\circlearrowleft"] & G\times G \arrow[d, "m"] \\
G\times G \arrow[r, "m"'] & G
\end{tikzcd}
,\quad
\begin{tikzcd}
pt\times G \arrow[r, "e\times id"] \arrow[dr, "\cong"'] & G\times G \arrow[d, "m"] & G\times pt \arrow[l, "id\times e"'] \arrow[dl, "\cong"] \\
 & G &
\end{tikzcd}
,\quad
\begin{tikzcd}
G \arrow[r, "(i{,}id)"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & G\times G \arrow[d, "m"] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[l, "(id{,}i)"'] \arrow[d] \\
pt \arrow[r, "e"'] & G & pt \arrow[l, "e"]
\end{tikzcd}
$$

<u>Rmk.</u>: Any scheme X/k is uniquely determined by its functor of points

$$
\begin{array}{ccc}
Alg_k & \xrightarrow{h_X} & Set \\
R & \longmapsto & \underbrace{Hom_k(\operatorname{Spec} R, X)}
\end{array}
$$

if X is affine, $X = \operatorname{Spec} A$,
this is $Hom_k(A, R)$

Moreover, for $X, Y/k$, a map $X \xrightarrow{f} Y$ is the same as a natural transformation $h_X \to h_Y$.

<u>Def.</u> (functorial): An <u>affine algebraic group</u> is an affine scheme $G = \operatorname{Spec} A$ together with natural maps of sets

$$
\forall R \in Alg_k : \quad
\begin{array}{rccc}
m_R : & G(R)\times G(R) & \to & G(R) \\
e_R : & \{*\} & \longrightarrow & G(R) \\
i_R : & G(R) & \longrightarrow & G(R)
\end{array}
$$

satisfying the usual group axioms.

<u>Rmk.</u>: Having an inverse is a property. (If it exists, it is uniquely determined.)

<u>Def.</u> (algebraic): A <u>(commutative) Hopf algebra</u> is a (commutative) k-algebra A together with algebra homomorphisms

$$
\begin{array}{lcll}
\Delta : A & \longrightarrow & A\otimes A & \text{(comultiplication)} \\
\varepsilon : A & \longrightarrow & k & \text{(counit)} \\
\iota : A & \longrightarrow & A & \text{(antipode)}
\end{array}
$$

such that the following diagrams commute:

$$
\begin{tikzcd}
A\otimes A\otimes A & A\otimes A \arrow[l, "\Delta\otimes id"'] \\
A\otimes A \arrow[u, "id\otimes\Delta"] \arrow[ur, phantom, "\circlearrowright"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
,\quad
\begin{tikzcd}
A & A\otimes A \arrow[l, "\varepsilon\otimes id"'] \arrow[r, "id\otimes\varepsilon"] & A \\
 & A \arrow[ul, equal] \arrow[u, "\Delta"] \arrow[ur, equal] &
\end{tikzcd}
,\quad
\begin{tikzcd}
A & A\otimes A \arrow[l, "(\iota\otimes id)"'] \arrow[r, "(id\otimes\iota)"] & A \\
k \arrow[u] \arrow[ur, phantom, "\circlearrowright"] & A \arrow[l, "\varepsilon"] \arrow[u, "\Delta"] \arrow[r, "\varepsilon"'] \arrow[ur, phantom, "\circlearrowleft"] & k \arrow[u]
\end{tikzcd}
$$

(coassociativity) (counit axiom) (coinverse axiom)

<u>Proposition</u>

There is an antiequivalence

$$
\begin{array}{ccc}
\{\text{affine algebraic groups}\} & \longleftrightarrow & \{\text{commutative Hopf algebras}\} \\
\operatorname{Spec} A & \longleftarrow & A \\
G & \longmapsto & \Gamma(G, \mathcal{O})
\end{array}
$$

Examples

(1) the additive group $\mathbb{G}_a$: $R \longmapsto (R,+)$. $\quad \mathbb{G}_a = \operatorname{Spec} k[x]$,

because $\operatorname{Hom}_k(k[x],R) \cong R$ compatibly with addition.

The Hopf algebra structure has
$$\Delta: k[x] \longrightarrow k[x]\otimes k[x] \cong k[y,z]$$
$$x \longmapsto y+z$$
$$\varepsilon: k[x] \to k, \quad x \mapsto 0$$
$$\imath: k[x] \to k[x], \quad x \mapsto -x.$$

(2) $\mathbb{G}_m$, multiplicative group: $R \longmapsto (R^\times,\cdot)$. $\quad \mathbb{G}_m = \operatorname{Spec} k[t,t^{-1}]$.

Indeed, $\operatorname{Hom}_k(k[t^{\pm1}],R) \cong R^\times$ compatibly with multiplication.

Hopf algebra structure:
$$\Delta: k[t^{\pm 1}] \longrightarrow k[u^{\pm1},v^{\pm1}]$$
$$t \longmapsto uv$$
$$\varepsilon: k[t^{\pm1}] \to k, \quad t \mapsto 1$$
$$\imath: k[t^{\pm1}] \to k[t^{\pm1}], \quad t \mapsto t^{-1}.$$

(3) GL_n: $R \longmapsto (\{n\times n \text{ invertible matrices over } R\}, \text{matrix multiplication})$.

$GL_n = \operatorname{Spec} A$ for $A = k[x_{ij}, \det^{-1}]_{1\le i,j\le n}$.

Hopf algebra has
$$\Delta: x_{ij} \longmapsto \sum_e x_{ie}\otimes x_{ej}$$
$$\varepsilon: x_{ij} \mapsto \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j\end{cases}$$
[and $\imath$ given by Cramer's rule].

More flexibly, if V is a k-vector space we can define
$$GL_V : \mathrm{Alg}_k \to \mathrm{Grp}$$
$$R \longmapsto (\operatorname{Aut}_R(V\otimes R), \circ)$$

If V is finite-dimensional, this gives back GL_n [without explicit coordinates].

(4) <u>Abstract groups</u>: Let H be any group and define $G := \coprod_{h\in H} \underbrace{pt_h}_{\text{i.e. pt, labelled by } h}$.

If H finite, then $G = \operatorname{Spec} A$ where $A = \prod_{h\in H} k_h$. On it, we always have
$$m: G\times G \to G, \quad (pt_h, pt_g) \mapsto pt_{hg}$$
$$e: pt = pt_1 \hookrightarrow G.$$

On the functor of points, this is $R \longmapsto \prod_{\pi_0(\operatorname{Spec} R)} H$.

(5) $n \in \mathbb{N}$, μ_n = roots of unity : $R \mapsto (\{r \in R \mid r^n = 1\}, \cdot)$.

$\mu_n = \operatorname{Spec}(k[t]/(t^n - 1))$.

Hopf algebra structure comes from that of $\mathbb{G}_m$. In fact, it is a sub-group scheme of $\mathbb{G}_m$.

(6) Assume char $k = p > 0$. Define $\alpha_p : R \to (\{r \in R \mid r^p = 0\}, +)$.

This is represented by $\operatorname{Spec}(k[x]/(x^p))$. The Hopf algebra structure comes from that of $\mathbb{G}_a$, α_p is a sub-group scheme of $\mathbb{G}_a$.

12. 3. 2025

Today: representations

G affine group scheme $/k$, A the associated commutative Hopf algebra

Def.: An algebraic representation of G on a vector space $V \in \mathrm{Vect}_k$ is a morphism of group functors $\psi : G \to GL_V$.

That is, for every $R \in \mathrm{Alg}_k$ it consists of a group homomorphism

$$\psi_R : G(R) \to GL_V(R) := \mathrm{Aut}_{\mathrm{Mod}_R}(V \otimes R),$$

functorially in R.

Remark: If $\dim V = n < \infty$, this amounts to a homomorphism of group schemes

$$\psi : G \to GL_V,$$

i.e. to a map of Hopf algebras

$$k[x_{ji}, \det^{-1}]_{1 \le i,j \le n} \to A,$$

i.e. to a matrix $(a_{ji})_{1 \le i,j \le n}$ of elements of A (which is invertible) s.t.

[This follows from $\circlearrowleft$.]

$$\Delta(a_{ji}) = \sum_k a_{jk} \otimes a_{ki}$$

$$\varepsilon(a_{ji}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Def.: $\mathrm{Rep}_k(G) :=$ {category of representations of G on finite-dim. vector spaces}.

Def.: A coalgebra is a k-vector space A together with k-linear maps $\Delta : A \to A \otimes A$, $\varepsilon : A \to k$ s.t. the coassociativity & counit axioms hold.

Def.: A (right) comodule of a coalgebra A is a vector space V together with a k-linear map $\rho : V \to V \otimes A$ s.t. the following diagrams commute:

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"'] & V\otimes A \arrow[d, "\rho\otimes id"] \\ V\otimes A \arrow[r, "id\otimes\Delta"'] & V\otimes A\otimes A \end{tikzcd} \quad , \quad \begin{tikzcd} V \arrow[r, "\rho"] \arrow[dr, "\text{"}id\text{"}"'] & V\otimes A \arrow[d, "id\otimes\varepsilon"] \\ & V\otimes k \end{tikzcd}$$

Remark: Pick a basis $\{e_i\}_{i\in I}$ of $V \in \mathrm{Vect}_k$ (possibly infinite).
Then, express ρ as $\rho(e_i) =: \sum_j e_j \otimes a_{ji}$. This yields a matrix $(a_{ji})_{j,i\in I}$ and we want to express the comodule axioms in terms of it.

The first:

$$e_i \overset{\rho}{\mapsto} \sum_j e_j \otimes a_{ji} \quad \begin{array}{l} \xmapsto{\rho\otimes id} \sum_{k,j} e_k \otimes a_{kj} \otimes a_{ji} \\ \qquad\qquad \| ! \\ \xmapsto{id\otimes\Delta} \sum_j e_j \otimes \Delta(a_{ji}) \end{array}$$

The second:

$$e_i \overset{\rho}{\mapsto} \sum_j e_j \otimes a_{ji} \xmapsto[\& \, V\otimes k \cong V]{id\otimes\varepsilon} \sum_j \varepsilon(a_{ji})\, e_j \overset{!}{=} e_j$$

Comparing coefficients, what we get is

$$\Delta(a_{ji}) = \sum_k a_{jk} \otimes a_{ki} \qquad \forall i,j \qquad (*)$$

$$\varepsilon(a_{ji}) = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases} \qquad \forall i,j \qquad (**)$$

Proposition (representations & coordinates)
Let G be an affine group scheme with Hopf algebra A. Let $V\in \mathrm{Vect}_k$. Then we have a canonical bijection

$$\{\text{representations of } G \text{ on } V\} \leftrightarrow \{A\text{-comodule structures on } V\}.$$

Corollary: $\mathrm{Rep}_k(G) \simeq \mathrm{Comod}_A^{fd}$.

Proof of proposition
Let $\psi: G \to GL_V$.

(1) $\forall R: \; G(R) \to GL_V(R) \subseteq \mathrm{End}_V(R)$ functorial group homomorphism.

Take
$$\begin{array}{ccccccc} \mathrm{Hom}_{Alg_k}(A,A) & = & G(A) & \longrightarrow & GL_V(A) \subseteq \mathrm{End}(V\otimes A, V\otimes A) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ id & \triangleq & u & \text{universal element} \longmapsto & \psi_u \end{array}$$

(2) We now have $\psi_u \in \mathrm{Hom}_{Mod_A}(V\otimes A, V\otimes A)$, from which we obtain a unique k-linear
$$\rho: V \cong V\otimes k \longrightarrow V\otimes A \xrightarrow{\psi_u} V\otimes A.$$

(3) Now we have $\rho \in \mathrm{Hom}_{\mathrm{Vect}_k}(V, V \otimes A)$. Each step can be reversed, i.e., from ρ we could recover ψ.

It remains to check:

ψ preserves composition & unit $\Leftrightarrow$ ρ satisfies comodule axioms.

Fix a basis of V.

Given a point $g \in G(R) \hat{=} f \in \mathrm{Hom}_{\mathrm{Alg}_k}(A, R)$, $\psi(g) = \underbrace{(f(a_{ji}))}_{\text{via the fixed basis}} \in \mathrm{Hom}_{\mathrm{Mat}_R}(V \otimes R, V \otimes R)$.

From the identities $(*)$ & $(**)$ [on p.6] we get:

- coefficient-wise, $\psi(g_1 \cdot g_2) = \psi(g_1) \circ \psi(g_2)$
- $1 \in G(R)$ acts by id. $\square$

<u>Def.</u>: Let A be a coalgebra, and take V to be A viewed as a vector space. Then $\rho := \Delta : A \to A \otimes A = V \otimes A$ is a comodule structure, called the regular corepresentation.

<u>Proposition</u> (finiteness)

A coalgebra, V comodule. Then

(1) every finite set $S \subseteq V$ lies in a fin.dim. subcomodule of V.

(2) every finite set $S \subseteq A$ lies in a fin.dim. subcoalgebra of A.

<u>Proof.</u> (1) Wlog. assume $S = \{v\}$. Choose a basis $\{a_i\}_{i \in I}$ of A (as a vector space). We can then express $\rho(v)$ as follows:

$$\rho(v) =: \sum_i v_i \otimes a_i \qquad \text{for some } v_1, \dots, v_n \in V$$

Moreover, we can similarly write

$$\rho(v_i) =: \sum_j v_{ij} \otimes a_j \qquad \text{for some } v_{ij} \in V.$$

By the comodule axiom(s), we find

$$\sum_{i,j} v_{ij} \otimes a_j \otimes a_i = \sum_i v_i \otimes \Delta(a_i).$$

It follows that each v_{ij} is a linear combination of the v_i!

$\Rightarrow k \cdot \{v, v_1, \dots, v_n\}$ is a subcomodule containing v.

(2) Again, wlog. $S = \{a\}$. By (1), there is a subcomodule $V \subseteq A$, $\dim V < \infty$, $a \in V$. Let $u_1, \dots, u_m$ be a basis of V.

Then $\Delta(u_i) =: \sum_j u_j \otimes a_{ji}$, $\Delta(a_{ji}) = \sum_k a_{jk} \otimes a_{ki}$ by $(*)$.

But this means that $\mathrm{span}_k\{u_j, a_{ji}\}$ is already a subcoalgebra.

<u>Note</u>: If A is a coalgebra, then $A^\vee = \mathrm{Hom}_{\mathrm{Vect}_k}(A,k)$ is an associative unital algebra

with $m: A^\vee \otimes A^\vee \to (A\otimes A)^\vee \xrightarrow{\Delta^\vee} A^\vee$ as multiplication and $\varepsilon^\vee$ as unit.

Conversely, if B is a <u>finite-dimensional</u> associative unital algebra, then $B^\vee$ becomes a coalgebra with $\Delta: B^\vee \xrightarrow{m^\vee} (B\otimes B)^\vee \underset{\text{f.d.}}{\xleftarrow{\cong}} B^\vee \otimes B^\vee$ etc.

Similarly, this translates an A-comodule V to an ~~7~~ $A^\vee$-module $V^\vee$.

<u>Prop.</u>: There is an equivalence of categories ~~for really~~

$$\{\text{fin. dim. coalgebras}\} \simeq \{\text{fin. dim. unital assoc. algebras}\}$$

as well as

$$\{\text{fin. gen. right } A\text{-comodules}\} \simeq \{\text{fin. gen. left } A^\vee\text{-modules}\}.$$

[Here A is f.d., so f.g. $\Leftrightarrow$ f.d.]

<u>Corollary</u> (comodule exact sequence)

A coalgebra, M comodule. Have exact sequence

$$0 \to M \xrightarrow{\rho} M\otimes A \xrightarrow{\rho\otimes\mathrm{id} - \mathrm{id}\otimes\Delta} M\otimes A\otimes A.$$

<u>Proof</u>: · ρ is injective by the second comodule axiom.

· The composition is zero by the first comodule axiom.

· It remains to prove: if $\sum m_i \otimes a_i$ is sent to zero, then it is in the image of ρ. Since this is an elementwise statement, we may pass to a comodule M over a coalgebra ~~B~~ A which are both finite-dimensional. [i.e., wlog. everything finite-dimensional.]

$\rightsquigarrow$ Now dualise, writing $B := A^\vee$, $N := M^\vee$, and consider the dual sequence

$$B\otimes B\otimes N \to B\otimes N \to N \to 0$$
$$b\otimes n \mapsto b\cdot n$$
$$b_1\otimes b_2 \otimes n \mapsto b_1\otimes b_2 n - b_1 b_2 \otimes n.$$

$\hookrightarrow$ It suffices to prove exactness (in the middle) for this sequence. But if $\sum_i b_i n_i = 0$, then it is the image of $-\sum_i 1 \otimes b_i \otimes n_i$ $\left[\mapsto + \sum_i b_i \otimes n_i - \sum 1 \otimes \underbrace{\sum_i b_i n_i}_{=0}\right]$. ✓ $\square$

Example (finite groups)

H abstract group, finite. $\rightsquigarrow$ $G = \coprod_{h \in H} \mathrm{pt}$ affine group scheme,

$$A = \prod_{h \in H} k_h \quad \text{its Hopf algebra.}$$

$\Delta : \prod_{h \in H} k_h \to \prod_{h', h'' \in H} k_{h', h''}$ takes $1_h \mapsto \sum_{\substack{h', h'', \\ h'h'' = h}} 1_{h', h''}$ and

ε is the projection to k_e.

Its dual $A^\vee$ is just the group algebra $k[H]$.

$\Rightarrow$ Finite-dim. G-reps ($\hat{=}$ f.d. A-comodules) are the same as finite-dimensional representations of H: $\mathrm{Rep}_k(G) \simeq \mathrm{Comod}_A^{f.d.} \simeq \mathrm{Mod}_{k[H]}^{fd}$.

Example (multiplicative group)

Recall: $\mathbb{G}_m$ has Hopf algebra $k[t^{\pm 1}]$, with $\Delta : k[t^{\pm 1}] \to k[y^{\pm 1}, z^{\pm 1}]$, $t \mapsto yz$

and $\varepsilon : k[t^{\pm 1}] \to k$, $t \mapsto 1$.

Now, a comodule V has $\rho : V \to V \otimes k[t^{\pm 1}]$

$v \mapsto \sum_i \rho_i(v) \otimes t^i$, a finite sum indexed by $\mathbb{Z}$ [or a Laurent poly in V].

The first comodule axiom yields $\sum_{i,j} \rho_j(\rho_i(v)) \otimes y^j z^i = \sum_i \rho_i(v) \otimes y^i z^i$,

from which we learn that $\rho_j(\rho_i(v)) = 0$ if $i \neq j$ and $\rho_i^2(v) = \rho_i(v)$.

Moreover, the second axiom implies $\sum_i \rho_i(v) = v$.

Altogether, we have linear maps $(\rho_i)_{i \in \mathbb{Z}}$ which form a complete set of orthogonal idempotents, so $V = \bigoplus_{i \in \mathbb{Z}} \rho_i(V)$. Hence,

$$\{\text{representations of } \mathbb{G}_m\} \leftrightarrow \{\mathbb{Z}\text{-graded vector spaces}\}$$

(and $\lambda \in \mathbb{G}_m(k)$ acts on the i-th graded piece by λ^i).

Example (additive group)

Recall: $\mathbb{G}_a$ has comultiplication $k[x] \to k[y,z]$, $x \mapsto y+z$ and counit $k[x] \to k$, $x \mapsto 0$.

A comodule structure on a v.s. V is
$$\rho : V \to V \otimes k[x], \quad v \mapsto \sum_{i \geq 0} \rho_i(v) \otimes x^i$$
for k-linear maps $\rho_i : V \to V$ subject to conditions. Namely:

$$V \to V \otimes k[x] \rightrightarrows V \otimes k[y,z]$$
$$v \mapsto \sum_{i \geq 0} \rho_i(v) \otimes x^i \longmapsto \sum_{i,j \geq 0} \rho_j(\rho_i(v)) \otimes y^j z^i$$
$$\longmapsto \sum_{\ell \geq 0} \rho_\ell(v) \otimes (y+z)^\ell$$

as well as
$$V \to V \otimes k[x] \to V$$
$$v \mapsto \sum_{i \geq 0} \rho_i(v) \otimes x^i \mapsto \rho_0(v) \overset{!}{=} v \quad \Rightarrow \rho_0 = \mathrm{id}_V.$$

From the first condition we get:
$$\forall i,j : \quad \rho_j(\rho_i(v)) = \binom{i+j}{i} \rho_{i+j}(v)$$

Note: The ρ_i commute, since we've seen that $\rho_j \circ \rho_i$ depends only on $i+j$.

Hence: ~~f.d.~~ Representations of $\mathbb{G}_a$ are the same as ~~f.d.~~ modules over the **divided power algebra** $B := k[\rho_1, \rho_2, \dots] / (\rho_i \rho_j - \binom{i+j}{i} \rho_{i+j})$ s.t. $\forall v \in V$ only finitely many $\rho_i(v)$ are nonzero. [local finiteness]

~~[The f.d. assumption can be removed by asking for local finiteness on the side of B.]~~

Claim: If $\operatorname{char} k = 0$, then $\operatorname{Rep}_k(\mathbb{G}_a) \simeq \{ (V, \phi) \mid V \in \mathrm{Vect}_k^{f.d.}, \ \phi : V \to V \text{ nilpotent} \}$

Indeed, let $s_i := \cancel{\rho_i / i!}$ $i! \cdot \rho_i \in B$ ~~[since char k = 0]~~. These satisfy $s_i \cdot s_j = s_{i+j}$,

from which we see $B \cong k[s_1]$. V being finite-dimensional means that local finiteness $\Leftrightarrow$ only finitely many ρ_i act as nonzero, so here some power of s_1 has to act by zero.

Claim: If $\operatorname{char} k > 0$, $\operatorname{char} k = p$, then

$$\operatorname{Rep}_k(\mathbb{G}_a) = \left\{ (V, \phi_1, \phi_2, \dots) \;\middle|\; \begin{array}{l} V \in \mathrm{Vect}_k^{fd}, \ \phi_i : V \to V \text{ endomorphism} \\ \text{s.t. } \phi_i^p = 0 \text{ and } \phi_i \phi_j = \phi_j \phi_i \ \forall i,j \\ \text{almost all zero} \end{array} \right\}.$$

__Lemma__ (Kummer)

If p is prime, the p-adic valuation of $\binom{n}{m}$ is given by the number of times we have to "carry over" when adding m and $(n-m)$ in base p.

__Proof:__ Write $n = \sum a_i p^i$, $m = \sum b_i p^i$, $(n-m) = \sum c_i p^i$. Then the quantity claimed to equal $\mathrm{val}_p \binom{n}{m}$ is $\dfrac{\sum b_i + \sum c_i - \sum a_i}{(p-1)}$. The valuations of the factorials we need are

$$\mathrm{val}_p(n!) = a_1 + (p a_2 + a_2) + (p^2 a_3 + p a_3 + a_3) + \cdots = \sum_i \frac{p^i - 1}{p-1} a_i$$

etc. It remains to observe

$$\sum_i \frac{p^i-1}{p-1} a_i - \sum_i \frac{p^i-1}{p-1} b_i - \sum_i \frac{p^i-1}{p-1} c_i = \frac{\sum b_i + \sum c_i - \sum a_i}{p-1} \quad . \quad //$$

Now let's prove the claim. First of all, we find that $\rho_i^p = 0 \quad \forall i \geq 1$, since $\rho_i^d = \frac{(2i)!}{i!\,i!} \frac{(3i)!}{i!\,(2i)!} \cdots \frac{(di)!}{((d-1)i)!\,i!} \rho_{di} = \frac{(di)!}{(i!)^d} \rho_{di}$.

In particular, $\rho_i^p = 0$ since when adding up $\underbrace{i + i + \cdots + i}_{p \text{ times}}$ in base p, we have to carry over at least once.

Also, B is graded with $\deg \rho_i = i$. For each $j \geq 1$, ρ_j can be expressed in terms of ρ_i with $i < j$ iff j is not a power of p; this again follows from the lemma. Hence, B is generated by the ρ_{p^d}, and each of them is not expressible by anything smaller.

$$\Rightarrow \quad B = k[\rho_1, \rho_p, \rho_{p^2}, \ldots] / (\rho_{p^d}^p = 0 \quad \forall d)$$

__19. 3. 2025__

__Today__: reconstruction of coalgebras (from) tensor categories

2. Reconstruction principles

Comodules and forgetful functors

__Note__: Let B be a finite-dimensional k-algebra and consider the forgetful functor

$$\omega : \mathrm{Mod}_B^{fg} \to \mathrm{Vect}_k^{fd}.$$

Then $B \cong \mathrm{Hom}(\omega, \omega)$, with addition & composition.

Indeed, each $b \in B$ gives a natural transformation

$$\eta : M \xrightarrow{b\cdot} M, \qquad M \in \mathrm{Mod}_B^{fd}.$$

On the other hand, we canonically have $B \in \mathrm{Mod}_B^{fg}$, so for any $\eta \in \mathrm{Hom}(\omega, \omega)$ there is the component $\eta_B : B \to B$, and $\eta_B(1) =: b \in B$. This already determines η by naturality:

$$\begin{tikzcd} \forall m & \forall M \arrow[r, "\eta_M"] \arrow[dr, phantom, "\circlearrowleft"] & M & m \\ 1 \arrow[u, mapsto] & B \arrow[u] \arrow[r, "\eta_B"'] & B \arrow[u] & 1 \arrow[u, mapsto] \\ & 1 \arrow[r, mapsto] & b & \end{tikzcd} \quad \rightsquigarrow \quad \eta_M(m) = \eta_B(1)\cdot m = b \cdot m.$$

For coalgebras, this works [for f.g $\to$ f.d] even without the finite-dimensionality assumption [on B].

$\to$ Let A be a coalgebra and consider

$$\omega : \mathrm{Comod}_A^{f.d.} \to \mathrm{Vect}_k^{f.d.}.$$

Given any $V \in \mathrm{Vect}_k$, not necessarily finite-dimensional, denote

$$\begin{aligned} \omega \otimes V : \mathrm{Comod}_A^{fd} &\to \mathrm{Vect}_k \\ M &\longmapsto \omega(M) \otimes V \end{aligned}$$

<u>Proposition</u> (comodules and forgetful functors)

The underlying vector space of A represents the functor $V \mapsto \mathrm{Hom}(\omega, \omega \otimes V)$.

That is, $\forall V \in \mathrm{Vect}_k$, there is a natural identification

$$\Psi : \mathrm{Hom}_k(A, V) \xleftrightarrow{\cong} \mathrm{Hom}(\omega, \omega \otimes V) : \Xi.$$

<u>Proof</u>: •) There is a natural morphism $\pi : \omega \to \omega \otimes A$ given by the comodule structure map: $\forall M \in \mathrm{Comod}_A^{fd}$: $M \xrightarrow{\rho} M \otimes A$, so we can map any $\phi : A \to V$ to

$$M \xrightarrow{\rho} M \otimes A \xrightarrow{\mathrm{id} \otimes \phi} M \otimes V,$$

giving us a morphism $\Psi(\phi) \in \mathrm{Hom}(\omega, \omega \otimes V)$.

•) For the other direction, consider A as a comodule over itself. It need not be finite-dimensional, but for each $a \in A$ we may fix a finite-dimensional subcomodule N of A containing a. Given $\eta \in \mathrm{Hom}(\omega, \omega \otimes V)$, let $\Xi(\eta) \in \mathrm{Hom}_k(A, V)$ map a to its image under $N \xrightarrow{\eta_N} N \otimes V \xrightarrow{\varepsilon \otimes \mathrm{id}} k \otimes V \cong V$. This is independent of the choice of N by naturality.

•) $\Xi \circ \Psi = \mathrm{id}$: given $\phi \in \mathrm{Hom}_k(A,V)$ and $a \in A$, $(\Xi \circ \Psi)(\phi)$ sends a to its image under $(N \xrightarrow{\Psi(\phi)} N \otimes V \xrightarrow{\varepsilon \otimes \mathrm{id}} k \otimes V \cong V)$, which unravels to

$$N \xrightarrow{\Delta_N} N \otimes A \xrightarrow{\mathrm{id} \otimes \phi} N \otimes V \xrightarrow{\varepsilon \otimes \mathrm{id}} k \otimes V \cong V,$$

but this can be rewritten to

$$\begin{tikzcd} N \arrow[r, hook] & A \arrow[r, "\Delta"] \arrow[rr, bend right, "\cong"'] & A \otimes A \arrow[r, "\varepsilon \otimes \mathrm{id}"] & k \otimes A \arrow[r, "\mathrm{id} \otimes \phi"] & k \otimes V \cong V, \end{tikzcd}$$

so a is really sent to $\phi(a)$. ✓

•) $\Psi \circ \Xi = \mathrm{id}$: fix $M \in \mathrm{Mod}_A^{fd}$ and $\eta \in \mathrm{Hom}(\omega, \omega \otimes V)$. By Prop. (finiteness), find a finite-dimensional subcoalgebra s.t. M is a comodule over B. We need to prove that η_M is given by

$$M \xrightarrow{\rho} M \otimes B \xrightarrow{\mathrm{id} \otimes \eta_B} M \otimes B \otimes V \xrightarrow{\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}} M \otimes k \otimes V \cong M \otimes V$$

Note: $M \otimes B$ is a B-comodule via $M \otimes B \xrightarrow{\mathrm{id} \otimes \Delta} M \otimes B \otimes B$, and this makes $M \xrightarrow{\rho} M \otimes B$ a B-comodule morphism:

"If you want to have a monkey, you need to pay for the bananas."

$$\begin{tikzcd} M \otimes B \arrow[r, "\mathrm{id} \otimes \Delta"] & M \otimes B \otimes B . \\ M \arrow[u, "\rho"] \arrow[r, "\rho"'] & M \otimes B \arrow[u, "\rho \otimes \mathrm{id}"'] \end{tikzcd}$$

Hence:

$$\begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d, "\eta_M"'] \arrow[dr, phantom, "\circlearrowleft"] & M \otimes B \arrow[d, "\eta_{M \otimes B}"] & & \\ M \otimes V \arrow[r, "\rho_M \otimes \mathrm{id}"'] \arrow[rrr, bend right, "="'] & M \otimes B \otimes V \arrow[r, "\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}"] & M \otimes k \otimes V \cong & M \otimes V, \end{tikzcd}$$

so $\eta_M = (M \xrightarrow{\rho_M} M \otimes B \xrightarrow{\eta_{M \otimes B}} M \otimes B \otimes V \xrightarrow{\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}} M \otimes k \otimes V \cong M \otimes V)$.

To finish, we note that $\eta_{M \otimes B} = \mathrm{id}_M \otimes \eta_B$ because $M \otimes B = \bigoplus_{i=1}^{\dim M} B$ as a B-comodule and η commutes with direct sums. ✓ $\square$

Corollary: Any k-coalgebra A is uniquely determined up to unique isomorphism by the category Comod_A^{fd} and $\omega : \mathrm{Comod}_A^{fd} \to \mathrm{Vect}_k^{fd}$.

Proof: The underlying vector space of A is determined by representing the functor $V \mapsto \mathrm{Hom}(\omega, \omega \otimes V)$. Moreover, as we have seen in the proof,

$\cdot)$ $$\mathrm{Hom}_k(A,A) \cong \mathrm{Hom}(\omega, \omega\otimes A)$$
$$\mathrm{id}_A \longleftrightarrow \pi, \quad \pi_M : M \xrightarrow{\rho_M} M\otimes A,$$

and (doing this twice!) $$\mathrm{Hom}_k(A, A\otimes A) \cong \mathrm{Hom}(\omega, \omega\otimes A\otimes A)$$
$$\Delta \longleftrightarrow (\pi\otimes \mathrm{id})\circ\pi, \quad M \xrightarrow{\rho} M\otimes A \underset{\mathrm{id}\otimes\Delta}{\overset{\rho\otimes \mathrm{id}}{\rightrightarrows}} M\otimes A\otimes A$$

$\cdot)$ and similarly $$\mathrm{Hom}(A,k) \cong \mathrm{Hom}(\omega, \omega\otimes k)$$
$$\varepsilon \longleftrightarrow \text{"id"}: \quad M \xrightarrow{\rho} M\otimes A \xrightarrow{\text{"id}\otimes\varepsilon\text{"}} M.$$

$\square$

<u>Remark:</u> variant – let A be a coalgebra and consider
$$\omega\otimes\omega : \mathrm{Comod}_A^{fd}\times \mathrm{Comod}_A^{fd} \to \mathrm{Vect}_k^{fd}$$
$$(M, N) \mapsto \omega(M)\otimes \omega(N).$$

Then $A\otimes A$ represents the functor $V\mapsto \mathrm{Hom}(\omega\otimes\omega, \omega\otimes\omega\otimes V)$,

i.e. there are natural bijections
$$\mathrm{Hom}_k(A\otimes A, V) \cong \mathrm{Hom}(\omega\otimes\omega, \omega\otimes\omega\otimes V).$$

One direction is just taking $(\phi_1,\phi_2)\in \mathrm{Hom}_k(A\otimes A, V)$ to:
$$\forall M,N: \quad M\otimes N \xrightarrow{\rho_M\otimes\rho_N} M\otimes A\otimes N\otimes A \xrightarrow[\&\text{ a flip!}]{\mathrm{id}_M\otimes \mathrm{id}_N\otimes(\phi_1,\phi_2)} M\otimes N\otimes V.$$

<u>Now:</u> If A is a commutative Hopf algebra, $G = \mathrm{Spec}\,A$, there is additional structure on $\mathrm{Comod}_A^{fd} \cong \mathrm{Rep}_k(G)$: We have

$\otimes$: $\mathrm{Rep}_k(G)\times\mathrm{Rep}_k(G) \to \mathrm{Rep}_k(G)$

$\mathbb{1}$: trivial rep.

$(-)^\vee$: $\mathrm{Rep}_k(G)^{op} \to \mathrm{Rep}_k(G)$.

We want to reconstruct m, e, S on A from this.

<u>Multiplication</u>: Let A be a coalgebra, and suppose we're given a linear map $m: A\otimes A \to A$. Then one can check that

$$\left.\begin{array}{l} \Delta: A \to A\otimes A \\ \varepsilon: A \to k \end{array} \quad \begin{array}{c}\text{are algebra maps}\\ \text{w.r.t. } m\end{array}\right\} \Leftrightarrow m \text{ is a coalgebra map:}$$

just read the following two diagrams:

$$\begin{tikzcd} A\otimes A \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "m"'] & A\otimes A\otimes A\otimes A \arrow[d, "m\otimes m \text{ (mb. w/ a flip)}"] \\ A \arrow[r, "\Delta"'] & A\otimes A \end{tikzcd}, \qquad \begin{tikzcd} A\otimes A \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"'] & k\otimes k \arrow[d, "m"] \\ A \arrow[r, "\varepsilon"'] & k \end{tikzcd}$$

If so, then we obtain
$$\otimes: \mathrm{Comod}_A^{fd} \times \mathrm{Comod}_A^{fd} \to \mathrm{Comod}_A^{fd}$$
$$(M, N) \longmapsto M\otimes N$$
with comodule structure $M\otimes N \xrightarrow{\rho_M\otimes\rho_N} M\otimes A\otimes N\otimes A \xrightarrow{\text{flip}} M\otimes N\otimes A\otimes A \xrightarrow{id\otimes m} M\otimes N\otimes A$

<u>Corollary</u>: Let A be a ~~Hopf~~ coalgebra with m as above [i.e. a coalg. map $A\otimes A \to A$].

(i) The multiplication $m: A\otimes A \to A$ is determined by Comod_A^{fd}, $\omega: \mathrm{Comod}_A^{fd} \to \mathrm{Vect}_k^{fd}$ and the tensor product $\otimes = \overset{m}{\otimes}$ [i.e. defined via m] on Comod_A^{fd}.

(ii) It is commutative if and only if the natural isomorphism

$$\omega(M)\otimes\omega(N) \overset{\sigma}{\xrightarrow{\sim}} \omega(N)\otimes\omega(M), \quad M, N \in \mathrm{Comod}_A^{fd}$$
$$m\otimes n \longleftrightarrow n\otimes m$$

comes from a natural isomorphism in Comod_A^{fd}.

(iii) It is associative if and only if the natural isomorphism

$$(\omega(M)\otimes\omega(N))\otimes\omega(P) \xrightarrow{\sim} \omega(M)\otimes(\omega(N)\otimes\omega(P)), \quad M, N, P \in \mathrm{Comod}_A^{fd}$$

comes from a natural isomorphism in Comod_A^{fd}.

<u>Proof</u>: (i) The map $m: A\otimes A \to A$ can be recovered from

$$\omega(M)\otimes\omega(N) \xrightarrow{\sim} \omega(M\overset{m}{\otimes}N) \xrightarrow{\rho} \omega(M\overset{m}{\otimes}N)\otimes A \xrightarrow{\sim} \omega(M)\otimes\omega(N)\otimes A$$

using the remark above [p. 14]: $\mathrm{Hom}_k(A\otimes A, V) \cong \mathrm{Hom}(\omega\otimes\omega, \omega\otimes\omega\otimes V)$.

(ii) m is commutative iff $A\otimes A \xrightarrow{m} A$ & $A\otimes A \xrightarrow{m\circ\sigma} A$ are equal. This is the case iff, for all $M, N \in \mathrm{Comod}_A^{fd}$, the corresponding maps $\omega(M \overset{m}{\otimes} N) \longrightarrow \omega(M \overset{m}{\otimes} N) \otimes A$ agree.

$\omega(M \overset{m\sigma}{\otimes} N) \longrightarrow \omega(M \overset{m\sigma}{\otimes} N) \otimes A$

Unravelling definitions / constructions, this is the case iff

$$\begin{tikzcd}
M\otimes N \arrow[r, "\rho\otimes\rho"] \arrow[d, leftrightarrow, "\sigma"'] & M\otimes A\otimes N\otimes A \arrow[r] & M\otimes N\otimes A\otimes A \arrow[r, "id\otimes id\otimes m"] & M\otimes N\otimes A \arrow[d, leftrightarrow, "\sigma\otimes id"] \\
N\otimes M \arrow[r, "\rho\otimes\rho"] & N\otimes A\otimes M\otimes A \arrow[r] & N\otimes M\otimes A\otimes A \arrow[r, "id\otimes id\otimes m"] & N\otimes M\otimes A
\end{tikzcd}$$

commutes. But this just expresses that/whether $\sigma : M\otimes N \to N\otimes M$ is a comodule map! ✓

(iii) is similar with more ω's. $\square$

<u>Unit</u>: Let now A be a coalgebra with a compatible multiplication.

<u>Corollary</u>: An element $e\in A$ is a unit for m compatibly with the coalgebra structure if and only if the corresponding $e: k\to A$ is a comodule structure on k, and for every $M\in \mathrm{Comod}^{fd}_A$ the obvious maps

$$k\otimes\omega(M) \cong \omega(M) \cong \omega(M)\otimes k$$

come from (natural) comodule isomorphisms.

<u>Proof</u>: ·) Compatibility means that Δ should preserve e, i.e. $\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \\ k \arrow[u, "e"] \arrow[r, "="] & k \arrow[u, "e\otimes e"'] \end{tikzcd}$.

Equivalently, $\begin{tikzcd} A \arrow[r, "\text{``}e\otimes id\text{''}"] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \\ k \arrow[u, "e"] \arrow[r, "e"] & A \arrow[u, "\Delta"'] \end{tikzcd}$ [flip this to see it ↖↘],

which expresses that k be a comodule via e.

·) Consider, $\forall M$,

$$M \cong M\otimes k \xrightarrow{\rho_M\otimes e} M\otimes A\otimes A \xrightarrow{id\otimes m} M\otimes A.$$

Under the Proposition, this corresponds to $(a\mapsto m(a\otimes e))\in \mathrm{Hom}_k(A,A)$, and e is a right unit iff that latter map is equal to id_A, hence iff the following commutes:

$$\begin{tikzcd}
M \otimes k \arrow[r, "\rho_M \otimes e"] & M \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes m"] & M \otimes A \\
M \arrow[u, "\text{"id"}"] \arrow[rr, "\rho_M"'] & & M \otimes A \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

But this commutes iff "id" [on the left] is a map of comodules.

Left unitality is analogous. $\square$

Antipode: Assume that A, a coalgebra, already has compatible m, e. If it also has an antipode $\iota : A \to A$, making it a Hopf algebra, then we get

$$\begin{tikzcd}
M \in \mathrm{Comod}_A^{fd} \arrow[r] \arrow[rr, mapsto, bend right] & \mathrm{Comod}_A^{fd} & \ni M^\vee := \mathrm{Hom}_k(M, k)
\end{tikzcd}$$

with the comodule structure

$$M^\vee \xrightarrow{\rho_{M^\vee}} M^\vee \otimes A \cong \mathrm{Hom}_k(M, A)$$

$$\phi \longmapsto \left(M \xrightarrow{\rho_M} M \otimes A \xrightarrow{\phi \otimes \mathrm{id}} k \otimes A \xrightarrow{\mathrm{id} \otimes \iota} \underset{\cong A}{k \otimes A}\right)$$

and thus a functor $(-)^\vee : \left(\mathrm{Comod}_A^{fd}\right)^{op} \to \mathrm{Comod}_A^{fd}$ lifting the usual duality on Vect_k^{fd}. Moreover, the maps

$$\underset{\text{(coevaluation)}}{\delta : k \longrightarrow M^\vee \overset{m}{\otimes} M} \qquad \& \qquad \underset{\text{(evaluation)}}{\varepsilon : M^\vee \overset{m}{\otimes} M \to k}$$

are comodule maps.

> ⚠ On this page, the order of $M^\vee$ & M may be wrong in some places. The ideas are ok though.

Proof idea: Either check this from comodule axioms or appeal to $\mathrm{Comod}_A^{fd} \simeq \mathrm{Rep}_k(G)$. $\square$

Corollary: Let A be a coalgebra equipped with compatible m, e [so a bialgebra]. Assume that $\forall M \in \mathrm{Comod}_A^{fd}$, the dual vector space $M^\vee$ has a natural [or functorial] A-comodule structure such that $\delta : k \to M^\vee \overset{m}{\otimes} M$ & $\varepsilon : M^\vee \overset{m}{\otimes} M \to k$ are comodule maps. Then there exists $\iota : A \to A$ making A into a Hopf algebra. [In other words, A then is a Hopf algebra.]

Proof idea: By Prop., $\mathrm{Hom}_k(A, A) \leftrightarrow \mathrm{Hom}(\omega, \omega \otimes A)$. Then for $M \in \mathrm{Comod}_A^{fd}$,
$\iota \leftrightarrow: \eta$

we get

$$\eta_M : \omega(M) \xrightarrow{\mathrm{id} \otimes \delta} \omega(M) \otimes \omega(M)^\vee \otimes \omega(M) \xrightarrow{\mathrm{id} \otimes \rho_{M^\vee} \otimes \mathrm{id}} \omega(M) \otimes \omega(M^\vee) \otimes A \otimes \omega(M)$$

$$\xrightarrow{\varepsilon \otimes \mathrm{id}} A \otimes \omega(M) \cong \omega(M) \otimes A.$$

[For details, see Szamuely Prop. 6.2.7.]

Last time: coalgebra A determined by Comod^{fd}_A, ω
Hopf algebra A determined by Comod^{fd}_A, ω, $\otimes$

Today: tensor categories, rigidity
algebraic Tannaka–Krein theorem

Tensor categories

$\mathcal{C}$ category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ functor.

- An associativity constraint for $\otimes$ is a natural isomorphism (of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$) $\quad \phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, $\quad X, Y, Z \in \mathcal{C}$

  such that the pentagon diagram commutes: $\forall X, Y, Z, W \in \mathcal{C}$:

$$\begin{tikzcd}
 & X \otimes (Y \otimes (Z \otimes W)) \arrow[dl, "\mathrm{id} \otimes \phi"'] \arrow[dr, "\phi"] & \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[d, "\phi"'] \arrow[rr, phantom, "\circlearrowleft"] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[d, "\phi"] \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[rr, "\phi \otimes \mathrm{id}"'] & & ((X \otimes Y) \otimes Z) \otimes W
\end{tikzcd}$$

- A commutativity constraint for $\otimes$ is a natural transformation

$$\psi_{X,Y} : X \otimes Y \to Y \otimes X, \qquad X, Y \in \mathcal{C}$$

  such that $\quad \psi_{Y,X} \circ \psi_{X,Y} = \mathrm{id}_{X \otimes Y}$. It is compatible with ϕ if the hexagon diagram commutes: $\forall X, Y, Z \in \mathcal{C}$:

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[dl, "\mathrm{id} \otimes \psi"'] \arrow[dr, "\phi"] & \\
X \otimes (Z \otimes Y) \arrow[d, "\phi"'] \arrow[rr, phantom, "\circlearrowleft"] & & (X \otimes Y) \otimes Z \arrow[d, "\psi"] \\
(X \otimes Z) \otimes Y \arrow[dr, "\psi \otimes \mathrm{id}"'] & & Z \otimes (X \otimes Y) \arrow[dl, "\phi"] \\
 & (Z \otimes X) \otimes Y &
\end{tikzcd}$$

- A unit object is an object $1 \in \mathcal{C}$ together with an isomorphism $\cancel{\nu :}\ 1 \xleftarrow{\simeq} 1 \otimes 1 : \nu$ such that the functors $\mathcal{C} \to \mathcal{C}$

$$X \mapsto 1 \otimes X, \qquad X \mapsto X \otimes 1$$

are fully faithful.

Def.: · A <u>monoidal category</u> is $(\mathcal{C}, \otimes, \Phi)$ as above which has a unit object.

· A <u>symmetric monoidal category</u> is $(\mathcal{C}, \otimes, \Phi, \Psi)$ as above which has a unit object. (This we'll also call a <u>tensor category</u>.)

Lemma

(1) There are natural isomorphisms $\alpha^1_X : 1 \otimes X \xrightarrow{\cong} X$ & $\beta : X \xrightarrow{\cong} X \otimes 1$ and the functors $1 \otimes (-)$, $(-) \otimes 1$ induce equivalences $\mathcal{C} \to \mathcal{C}$.

(2) A unit object $(1, \nu)$ is unique up to unique isomorphism.

Proof

(1) To construct $\alpha^1_X : 1 \otimes X \xrightarrow{\cong} X \in \mathrm{Hom}(1 \otimes X, X)$ we use that $X \mapsto 1 \otimes X$ is fully faithful:

$$\mathrm{Hom}(1 \otimes \underbrace{1 \otimes X}_{1 \otimes X}) \longleftrightarrow \mathrm{Hom}(1 \otimes X, X)$$

$$\begin{array}{ccc} \cup & & \\ \nu \otimes \mathrm{id}_X & \longleftrightarrow : & \alpha^1_X . \end{array}$$

By full faithfulness and the fact that $\nu \otimes \mathrm{id}_X$ is an iso, α^1_X is an iso. ✓

(2) Given unit objects $(1, \nu)$, $(1', \nu')$, consider

$$\gamma : 1 \underset{\beta^{1'}_1}{\overset{\cong}{\longleftarrow}} 1 \otimes 1' \underset{\alpha^1_{1'}}{\overset{\cong}{\longrightarrow}} 1',$$

which is the unique isomorphism making the following commute:

$$\begin{tikzcd} 1 \otimes 1 \arrow[r, "\gamma \otimes \gamma"] \arrow[d, "\nu"'] & 1' \otimes 1' \arrow[d, "\nu'"] \\ 1 \arrow[r, "\gamma"] & 1' \end{tikzcd}$$

[We don't prove this.]

//

Remark: $L \in \mathcal{C}$ is called <u>invertible</u> if $\begin{array}{c} \mathcal{C} \to \mathcal{C} \\ X \mapsto L \otimes X \end{array}$ is an equivalence.

[$\mathcal{C}$ assumed tensor cat.] In fact, there then exists an inverse $L' \in \mathcal{C}$ with $L \otimes L' \cong 1$.

Remark: • The pentagon axiom implies similar commutativity for all bracketings.
$\rightarrow$ We can ignore bracketings.

• The hexagon axiom implies that we can also ignore ordering.
$\rightsquigarrow$ Ignore brackets & order for multiple $\otimes$ products.

next page

Def (tensor functor)

Let $\mathcal{C}, \mathcal{C}'$ be tensor categories. A <u>tensor functor</u> $\mathcal{C} \to \mathcal{C}'$ is a pair (F, c) where

- $F : \mathcal{C} \to \mathcal{C}'$ is a functor, and
- c is a natural isomorphism of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}'$,
$$c_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\cong} F(X \otimes Y), \qquad X, Y \in \mathcal{C}$$

such that:

- $$\begin{tikzcd} F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r, "\mathrm{id} \otimes c"] \arrow[d, "\phi'"'] & F(X) \otimes F(Y \otimes Z) \arrow[r, "c"] \arrow[d, phantom, "\circlearrowright"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\phi)"] \\ (F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "c \otimes \mathrm{id}"'] & F(X \otimes Y) \otimes F(Z) \arrow[r, "c"'] & F((X \otimes Y) \otimes Z) \end{tikzcd} \qquad \forall X, Y, Z \in \mathcal{C},$$

- $$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "c"] \arrow[d, "\psi'"'] & F(X \otimes Y) \arrow[d, "F(\psi)"] \\ F(Y) \otimes F(X) \arrow[r, "c"'] & F(Y \otimes X) \end{tikzcd}$$

- $(1, \nu)$ is sent to a unit object of $\mathcal{C}'$.

Def. (morphism of tensor functors)

A <u>morphism of tensor functors</u> $(F, c), (G, d) : \mathcal{C} \to \mathcal{C}'$ is a natural transformation $\eta : F \to G$ such that

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "c"] \arrow[d, "\eta_X \otimes \eta_Y"'] \arrow[dr, phantom, "\circlearrowright"] & F(X \otimes Y) \arrow[d, "\eta_{X \otimes Y}"] \\ G(X) \otimes G(Y) \arrow[r, "d"'] & G(X \otimes Y) \end{tikzcd} \quad \& \quad \begin{tikzcd} F(1) \arrow[d, "\eta_1"'] \arrow[dr, phantom, "\circlearrowleft"] & 1' \arrow[l, "\cong"'] \arrow[d, "\mathrm{id}"] \\ G(1) & 1' \arrow[l, "\cong"] \end{tikzcd}.$$

Remark: This is compatible with extending $\otimes$ to finite families [as in the remark on p. 19].

Remark: A morphism η of tensor functors is an isomorphism iff η is an isomorphism of functors [i.e. forgetting the extra structure].

Remark: A tensor functor (F, c) is an equivalence [i.e. has an inverse equivalence which is a tensor functor etc.] iff the functor F is an equivalence.

Def: Given a tensor functors $(F, c), (G, d) : \mathcal{C} \to \mathcal{C}'$ we write $\mathrm{Hom}^{\otimes}(F, G)$ for the set of morphisms of tensor functors from (F, c) to (G, d).

Internal hom

Def.: Let $X, Y \in \mathcal{C}$ [$\mathcal{C}$ a tensor category]. If the functor

$$\mathcal{C}^{op} \to \mathrm{Set}, \qquad T \longmapsto \mathrm{Hom}(T \otimes X, Y)$$

is representable, we denote the ~~corresponding~~ representing object by $\underline{\mathrm{Hom}}(X,Y)$ and call it the [or an] internal hom.

Remark: If so, then by definition

$$\forall T \in \mathcal{C}: \quad \mathrm{Hom}(T \otimes X, Y) \text{ "="} \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X,Y)),$$

so in particular there is an evaluation map ε defined by

$$\mathrm{Hom}(\underline{\mathrm{Hom}}(X,Y) \otimes X, Y) = \mathrm{Hom}(\underline{\mathrm{Hom}}(X,Y), \underline{\mathrm{Hom}}(X,Y))$$

$$\varepsilon \longleftrightarrow \mathrm{id}.$$

Note: If $\underline{\mathrm{Hom}}(X,Y)$ exists, it is uniquely determined, as is the evaluation ε.

Remark

- $\mathrm{Hom}(1, \underline{\mathrm{Hom}}(X,Y)) = \mathrm{Hom}(X,Y)$
- $\underline{\mathrm{Hom}}(X,Y) \otimes \underline{\mathrm{Hom}}(Y,Z) \to \underline{\mathrm{Hom}}(X,Z)$

Def.: A dual of $X \in \mathcal{C}$ is an object $X^\vee \in \mathcal{C}$ together with

$$\delta: 1 \to X \otimes X^\vee, \qquad \varepsilon: X^\vee \otimes X \to 1,$$

called coevaluation & evaluation, such that

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[rrrr, bend right, "\mathrm{id}"'] & 1 \otimes X \arrow[r, "\delta \otimes \mathrm{id}"] & X \otimes X^\vee \otimes X \arrow[r, "\mathrm{id} \otimes \varepsilon"] & X \otimes 1 \arrow[r, "\cong"] & X \end{tikzcd}$$

&

$$\begin{tikzcd} X^\vee \arrow[r, "\cong"] \arrow[rrrr, bend right, "\mathrm{id}"'] & X^\vee \otimes 1 \arrow[r, "\mathrm{id} \otimes \delta"] & X^\vee \otimes X \otimes X^\vee \arrow[r, "\varepsilon \otimes \mathrm{id}"] & 1 \otimes X^\vee \arrow[r, "\cong"] & X^\vee \end{tikzcd}$$

Def.: A tensor category $\mathcal{C}$ is rigid if every object admits a dual.

Lemma: In a rigid tensor category, internal homs exist and are given by

$$\underline{\mathrm{Hom}}(X,Y) \cong X^\vee \otimes Y \cong Y \otimes X^\vee$$

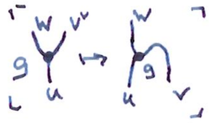
Proof: Need $\forall U, V, W \in \mathcal{C}$: $\quad \mathrm{Hom}(U \otimes V, W) = \mathrm{Hom}(U, W \otimes V^\vee)$.

To construct it, send $f \in \mathrm{Hom}(U \otimes V, W)$ to

$$\left(U \xrightarrow{\cong} U \otimes 1 \xrightarrow{\mathrm{id}\otimes\delta} U \otimes V \otimes V^\vee \xrightarrow{f \otimes \mathrm{id}} W \otimes V^\vee\right) \in \mathrm{Hom}(U, W \otimes V^\vee),$$

and for the inverse send $g \in \mathrm{Hom}(U, W \otimes V^\vee)$ to

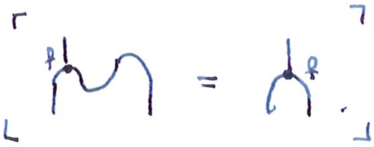
$$\left(U \otimes V \xrightarrow{g \otimes \mathrm{id}} W \otimes V^\vee \otimes V \xrightarrow{\mathrm{id}\otimes\varepsilon} W \otimes 1 \xrightarrow{\cong} W\right) \in \mathrm{Hom}(U \otimes V, W).$$



These really are mutual inverses: starting with f, we obtain

$$\begin{tikzcd}
U \otimes V \arrow[r, "\cong"] \arrow[d, equal] & U \otimes 1 \otimes V \arrow[r, "\mathrm{id}\otimes\delta\otimes\mathrm{id}"] \arrow[d, equal] & U \otimes V \otimes V^\vee \otimes V \arrow[r, "f\otimes\mathrm{id}\otimes\mathrm{id}"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & W \otimes V^\vee \otimes V \arrow[r, "\mathrm{id}\otimes\varepsilon"] & W \otimes 1 \arrow[r, "\cong"] \arrow[d, equal] & W \arrow[d, equal] \\
U \otimes V \arrow[r, "\cong"] & U \otimes 1 \otimes V \arrow[r, "\mathrm{id}\otimes\delta\otimes\mathrm{id}"] \arrow[d, equal] \arrow[drr, phantom, "\circlearrowright"] & U \otimes V \otimes V^\vee \otimes V \arrow[r, "\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon"] & U \otimes V \otimes 1 \arrow[r, "f\otimes\mathrm{id}"] \arrow[d, equal] & W \otimes 1 \arrow[r, "\cong"] & W \\
& U \otimes 1 \otimes V \arrow[rr, equal] & & U \otimes V \otimes 1 . & &
\end{tikzcd}$$

Other direction analogously.



Corollary: In a rigid tensor category,

$$X^\vee = \underline{\mathrm{Hom}}(X, 1) \qquad \forall X.$$

In particular, $X^\vee$ is unique up to isomorphism. If we additionally fix the evaluation $\varepsilon : X^\vee \otimes X \to X$, such an isomorphism is unique.

Proof: Previous lemma + Yoneda lemma.
([EGNO, Prop. 2.10.5] for diagrammatic proof)

Corollary: In a rigid tensor category $\mathcal{C}$

- every $X \in \mathcal{C}$ is <u>reflexive</u>, i.e. $X \xrightarrow{\cong} (X^\vee)^\vee$,
- $(-)^\vee$ commutes with $\otimes$,
- $\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \xrightarrow{\cong} \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 Y_2)$, and
- tensor functors commute with $(-)^\vee$ and $\underline{\mathrm{Hom}}(-,-)$.

⌜All of these are easy to see from the definition of $X^\vee$, but not obvious if we defined $X^\vee$ as $\underline{\mathrm{Hom}}(X, 1)$.⌟

Note: The assignments $X \mapsto X^\vee$ assembles into a contravariant functor $\mathcal{C} \to \mathcal{C}$ [i.e. into $\mathcal{C}^{op} \to \mathcal{C}$] as follows: Given $f : X \to Y$ we get for every $T \in \mathcal{C}$

$$\begin{array}{ccc}
\mathrm{Hom}(T \otimes Y, 1) & \longrightarrow & \mathrm{Hom}(T \otimes X, 1) \\
\shortparallel & & \shortparallel \\
\mathrm{Hom}(T, Y^\vee) & \longrightarrow & \mathrm{Hom}(T, X^\vee)
\end{array}$$

which ~~indu~~ corresponds to a morphism $f^\vee : Y^\vee \to X^\vee$ via the Yoneda lemma.

In terms of ε & δ, we can also write it as

$$f^\vee : Y^\vee \xrightarrow{\cong} Y^\vee \otimes 1 \xrightarrow{id \otimes \delta} Y^\vee \otimes X \otimes X^\vee \xrightarrow{id \otimes f \otimes id} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{\varepsilon \otimes id} 1 \otimes X^\vee \xrightarrow{\cong} X^\vee$$

[Doesn't matter but some orders should be changed!]

$\leadsto$ We have a tensor anti-equivalence $(-)^\vee$ on $\mathcal{C}$,
$(-)^\vee : \mathcal{C}^{op} \to \mathcal{C}$.

<u>Lemma</u> (rigidity of functors)

~~A~~ morphism of tensor functors between rigid tensor categories is an isomorphism.

<u>Proof</u>: Let $F, G : \mathcal{C} \to \mathcal{C}'$ be tensor functors, $\lambda : F \to G$ a morphism.

$\leadsto \forall X \in \mathcal{C}$ we have $\lambda_X : F(X) \to G(X)$, so in particular we have

$$\begin{array}{ccc} \lambda_{X^\vee} : F(X^\vee) & \to & G(X^\vee) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ F(X)^\vee & \to & G(X)^\vee, \end{array}$$

and can consider $\lambda_{X^\vee}$ as a morphism $F(X)^\vee \to G(X)^\vee$. Upon applying $(-)^\vee$, we get

$$\begin{array}{ccc} \lambda_{X^\vee}^\vee : G(X)^{\vee\vee} & \to & F(X)^{\vee\vee} \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ G(X) & & F(X). \end{array}$$

[apparently called "transpose"]

These assemble to a morphism of tensor functors $G \to F$, and we now want to check that this is pointwise inverse to λ. To that end, let $X \in \mathcal{C}$ and consider the diagram

$\cancel{G(X) \otimes F(X)^\vee \otimes F}$ $\quad \cancel{id \otimes \delta} \uparrow \quad \cancel{1 \otimes G(X)}$

$$\begin{tikzcd} F(X) \otimes F(X^\vee) \otimes G(X) \arrow[r, "id \otimes \lambda_{X^\vee}^\vee \otimes id"] \arrow[dr, "\lambda_X \otimes \lambda_{X^\vee} \otimes id"] & F(X) \otimes G(X^\vee) \otimes G(X) \arrow[r, "id \otimes \varepsilon"] \arrow[d, "\lambda_X \otimes id \otimes id"] \arrow[dr, phantom, "\circlearrowleft"] & F(X) \otimes 1 \arrow[d, "\lambda_X \otimes id"] \\ 1 \otimes G(X) \arrow[u, "\delta \otimes id"] \arrow[r, "\delta \otimes id"'] \arrow[rr, bend right, "id"'] & G(X) \otimes G(X^\vee) \otimes G(X) \arrow[r, "id \otimes \varepsilon"'] & G(X) \otimes 1 \end{tikzcd},$$

in which the path "$\uparrow \to \cdot \to \downarrow$" is "$\lambda_X \circ \lambda_{X^\vee}^\vee$". We can see that it commutes – the right square does because $\otimes$ is a functor. For the left square, naturality of λ & compatibility of δ with F & G show

$$\begin{tikzcd} 1 \arrow[r, "\delta"] \arrow[dr, phantom, "\circlearrowleft"] & F(X) \otimes F(X^\vee) \arrow[d, "\lambda_X \otimes \lambda_{X^\vee}"] \\ 1 \arrow[u, "id = \lambda_1"] \arrow[r, "\delta"] & G(X) \otimes G(X^\vee) \end{tikzcd}$$

and we only have to tensor with $G(X)$.

$\Rightarrow \lambda_X \circ \lambda_{X^\vee}^\vee = id$. Other direction similarly, or argue by duality $\square$

Examples

- G affine group scheme / field k. $\rightsquigarrow$ $\mathrm{Rep}_k(G)$ is rigid.
- For a commutative ring, Proj^{fg}_R – the  of finitely generated projective R-modules – is rigid.

2.4.2025

Last time
- tensor categories
- rigidity
- A morphism of tensor functors between rigid tensor categories is an isomorphism.

Today
- Algebraic Tannaka-Krein theorem
- neutral Tannakian categories and Tannakian reconstruction theorem

Correction: lecture 1, example 4

↳ H abstract group $\longrightarrow$ $G = \coprod_{h\in H} pt_h$ ~~abstr~~ group scheme.

If H is finite then $G = \mathrm{Spec}\,A$ for $A = \prod_{h\in H} k_h$.

[Otherwise, G is not quasicompact hence cannot be an affine scheme.]

The algebraic Tannaka-Krein theorem

G affine group scheme / field k. $(\mathrm{Rep}_k(G), \otimes)$ rigid tensor category.

$\omega : \mathrm{Rep}_k(G) \longrightarrow \mathrm{Vect}^{fd}_k$ tensor functor (forgetting rep. data)

(also ω^G)

↳ For each $R \in \mathrm{Alg}_k$, denote $\omega_R := \omega \otimes R : \mathrm{Rep}_k(G) \to \mathrm{Proj}^{fg}_R$

$$V \mapsto \omega(V) \otimes_k R$$

Notation: Consider the functors $\mathrm{Alg}_k \longrightarrow \mathrm{Set}$

$$\left.\begin{array}{lll} \mathrm{End}(\omega) : R & \longmapsto & \mathrm{Hom}(\omega_R, \omega_R) \text{ ... morphisms of functors} \\ \quad\cup\!\!\!| & & \\ \mathrm{End}^{\otimes}(\omega) : R & \longmapsto & \mathrm{Hom}^{\otimes}(\omega_R, \omega_R) \text{ ... morphisms of tensor functors} \end{array}\right\} \text{valued in monoids}$$

$$\left.\begin{array}{lll} \quad\cup\!\!\!| & & \\ \mathcal{A}ut^{\otimes}(\omega) : R & \longmapsto & \mathrm{Isom}^{\otimes}(\omega_R, \omega_R) \text{ ... tensor automorphisms} \end{array}\right\} \text{valued in groups}$$

// Note: In current setup, $\mathrm{End}^{\otimes}(\omega) = \mathcal{A}ut^{\otimes}(\omega)$! (by rigidity) //

Theorem (algebraic Tannaka-Krein)

There is a canonical isomorphism of group-valued functors

$$G \xrightarrow{\cong} \mathcal{A}ut^{\otimes}(\omega).$$

In particular, $\mathcal{A}ut^{\otimes}(\omega)$ is representable by an affine group scheme.

__Remarks__

- originally for topological groups
- Shows how to conveniently reconstruct G from $(\mathrm{Rep}_k(G), \otimes, \omega^G)$.

__Proof__ Write A for the commutative Hopf algebra corresponding to G.

·) The map [of group-valued functors] is given by

$$\begin{array}{ccc} G(R) & \longrightarrow & \mathcal{A}ut^{\otimes}(\omega \otimes R) \\ \rotatebox{90}{$\in$} & & \\ g & \longmapsto & (V \otimes R \xrightarrow{g} V \otimes R,\ V \in \mathrm{Rep}_k(G)) \end{array} \quad , \quad R \in \mathcal{A}lg_k .$$

·) "Elements of ~~$\mathcal{A}ut^{\otimes}(\omega)$~~ $\mathcal{E}nd(\omega)(R)$ correspond to k-linear maps $A \to R$".

↳ $$\mathcal{E}nd(\omega)(R) \overset{\text{def}}{=} \mathrm{Hom}_R(\omega \otimes R, \omega \otimes R) \cong \mathrm{Hom}_k(\omega, \omega \otimes R) \underset{\uparrow}{=} \mathrm{Hom}_{\mathrm{Vect}_k}(A, R)$$

(cf. comodules and forgetful functors)

·) "An element of $\mathcal{E}nd(\omega)(R)$ lies in $\mathcal{E}nd^{\otimes}(\omega)(R)$ iff the corresponding $A \to R$ is an algebra map."

↳ $\eta_R : \omega \otimes R \to \omega \otimes R$ is a map of tensor functors iff

$$\begin{tikzcd}
\omega(\cdot \otimes \cdot) \otimes R \arrow[r, "\eta_R"] & \omega(\cdot \otimes \cdot) \otimes R \\
(\omega(\cdot) \otimes R) \otimes_R (\omega(\cdot) \otimes R) \arrow[u, "c"', "\cong"] \arrow[r, "\eta_R \otimes_R \eta_R"] \arrow[ur, phantom, "\circlearrowleft"] \arrow[d, equal] & (\omega(\cdot) \otimes R) \otimes_R (\omega(\cdot) \otimes R) \arrow[u, "c"', "\cong"] \arrow[d, equal] \\
\omega(\cdot) \otimes \omega(\cdot) \otimes R & \omega(\cdot) \otimes \omega(\cdot) \otimes R
\end{tikzcd}$$

Now, when identifying $\mathrm{Hom}_{\mathrm{Vect}_k}(A, R) \cong \mathrm{Hom}(\omega, \omega \otimes R)$, we also identify

$$\lambda : A \to R \quad \hat{=} \quad \eta_R$$

$$\mathrm{Hom}_{\mathrm{Vect}_k}(A \otimes A, R) \cong \mathrm{Hom}(\omega \otimes \omega, \omega \otimes \omega \otimes R)$$

$$A \otimes A \xrightarrow{\lambda \otimes \lambda} R \otimes R \xrightarrow{m} R \quad \hat{=} \quad \eta_R \otimes_R \eta_R$$

$$A \otimes A \xrightarrow{m} A \xrightarrow{\lambda} R \quad \hat{=} \quad c^{-1} \circ \eta_R \circ c \ ,$$

so commutativity of the diagram above corresponds to λ being an algebra map.

·) By the Lemma (rigidity of functors) we have $\mathcal{A}ut^{\otimes}(\omega) = \mathcal{E}nd^{\otimes}(\omega)$, where we use that $\mathrm{Rep}_k(G)$ and $\mathrm{Proj}^{\mathrm{fg}}_R$ are rigid.

·) Altogether, we have exhibited a map $G \to \mathcal{A}ut^{\otimes}(\omega)$ and seen that for every R there is an isomorphism $G(R) = \mathrm{Hom}_k(A, R) \xrightarrow{\cong} \mathcal{E}nd^{\otimes}(\omega)$.

$$\Rightarrow \forall R: \quad \begin{tikzcd} G(R) \arrow[r] \arrow[rr, bend right, "\cong"'] & \mathcal{A}ut^{\otimes}(\omega)(R) \arrow[r, "="] & \mathcal{E}nd^{\otimes}(\omega)(R) \end{tikzcd}$$

$\square$

What about group homomorphisms?

- Let H, G be affine group schemes $/ k$. Given a homomorphism $f: H \to G$, we get

$$\mathrm{Rep}_k(H) \xleftarrow{f^*} \mathrm{Rep}_k(G)$$
$$(H \xrightarrow{f} G \xrightarrow{\psi} GL_V) \longleftarrow\!\shortmid (\psi: G \to GL_V),$$

and clearly $\omega^H \circ f = \omega^G$.

<u>Corollary</u>: H, G affine group schemes $/ k$, $F: \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(H)$ tensor functor such that

$$\begin{tikzcd} \mathrm{Rep}_k(H) \arrow[dr, "\omega^H"'] & & \mathrm{Rep}_k(G) \arrow[ll, "F"'] \arrow[dl, "\omega^G"] \arrow[dll, phantom, "\circlearrowleft"] \\ & \mathrm{Vect}_k^{fd} & \end{tikzcd}.$$

Then $\exists!$ group homomorphism $f: H \to G$ such that $F = f^*$.

This induces a canonical bijection

$$\{\text{group homomorphisms } H \to G\} \leftrightarrow \left\{\begin{array}{l}\text{tensor functors } \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(H) \\ \text{compatible with } \omega^G \ \& \ \omega^H\end{array}\right\}$$

$$f \longmapsto f^*$$

$$\omega^F : \longleftarrow\!\shortmid F$$

<u>Proof</u>: ·) We have already discussed $f \mapsto f^*$.

·) Conversely, given F, we get for each $R \in \mathrm{Alg}_k$:

$$\begin{tikzcd} \mathrm{Rep}_k(H) \arrow[dr, "\omega^H \otimes R"'] & & \mathrm{Rep}_k(G) \arrow[ll, "F"'] \arrow[dl, "\omega^G \otimes R"] \arrow[dll, phantom, "\circlearrowleft"] \\ & \mathrm{Proj}_R^{fg} & \end{tikzcd}.$$

This induces $\quad \mathcal{A}ut^{\otimes}(\omega^H) \xrightarrow{\omega^F} \mathcal{A}ut^{\otimes}(\omega^G)$

$\forall R: \quad \mathrm{Hom}(\omega^H \otimes R, \omega^H \otimes R) \to \mathrm{Hom}(\omega^G \otimes R, \omega^G \otimes R)$

$$\eta_R \longmapsto \text{"}\eta_R \circ F\text{"}$$

[that is, $\forall V$, $(\eta_R(V): V \otimes R \to V \otimes R) \mapsto (\eta_R(F(V)): F(V) \otimes R \to F(V) \otimes R)$]

but $\mathcal{A}ut^{\otimes}(\omega^H) \cong H$, $\mathcal{A}ut^{\otimes}(\omega^G) \cong G$.

One now checks that these constructions are inverse to each other. //

Remark: In particular,

$$\{\text{automorphisms of } G\} \longleftrightarrow \left\{ \text{tensor autom. } F \text{ s.t. } \begin{tikzcd} \mathrm{Rep}_k(G) \arrow[dr, "\omega^G"'] & & \mathrm{Rep}_k(G) \arrow[ll, "F"'] \arrow[dl, "\omega^G"] \\ & \mathrm{Vectfd}_k \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd} \right\}$$

Example (warning) ...... "Let us not fall asleep on the laurels".

Take the following finite groups over $\mathbb{C}$:

$$D \qquad \not\cong \qquad Q$$

dihedral group of order 8 [symmetries of $\square$] — unit quaternions.

"There is no point in doing the sudoku right now, but this is what you get out."

They are not isomorphic but they have the same character tables.

$\Rightarrow$ The Grothendieck ~~groups~~ rings are isomorphic: $K_0(D) \cong K_0(Q)$.

Why is this not a contradiction?

Answer: The associativity / commutativity constraints for the representation categories do not match, so that one cannot write a tensor isomorphism between them.

The main Tannakian reconstruction theorem

Can we characterise which categories $\mathcal{C}$ are of the form $\mathrm{Rep}_k(G)$?

Recall

- A category $\mathcal{C}$ is additive if its hom sets are $\mathbb{Z}$-modules, composition is $\mathbb{Z}$-bilinear, and finite products [= coproducts $\rightsquigarrow$ write "$\oplus$"] exist.

  It is R-linear – for a commutative ring R – if the hom spaces are $\mathbb{Z}$-modules and composition is R-bilinear.

- A functor is additive if it preserves $\oplus$, in particular the zero object [= empty $\oplus$].

- $\mathcal{C}$ is abelian if it is additive, has all kernels & cokernels, and images agree with coimages.

Def: An abelian tensor category is a tensor category $(\mathcal{C}, \otimes)$ where $\mathcal{C}$ is abelian and $\otimes$ is additive.

Remark: If $(\mathcal{C}, \otimes)$ is a rigid tensor category such that $\mathcal{C}$ is abelian, then the tensor functor $\otimes$ commutes with all limits and colimits, so in particular it is additive.

Proof: For $Y \in \mathcal{C}$, $\cdot \otimes Y$ has left and right adjoint $\cdot \otimes Y^\vee = \underline{\mathrm{Hom}}(Y, \cdot) : \mathcal{C} \to \mathcal{C}$. //

Remark Let $(\mathcal{C}, \otimes)$ be an abelian tensor category and define $R := \mathrm{End}_{\mathcal{C}}(1)$.

Then R is a commutative ring and $(\mathcal{C}, \otimes)$ is R-linear.

↳ Proof: R acts on each $\mathrm{Hom}_{\mathcal{C}}(X,Y) \cong \mathrm{Hom}_{\mathcal{C}}(1 \otimes X, 1 \otimes Y)$, and ~~so in particular~~ its action on $\mathrm{Hom}_{\mathcal{C}}(X,X)$ commutes with each endomorphism of X, for all X.
Setting $X = 1$ shows R to be commutative.

Notation: For $X \in \mathcal{C}$, $\mathcal{C}$ abelian, denote by $\langle X \rangle$ the smallest abelian full subcategory of $\mathcal{C}$ which contains X.

If $X \in \mathcal{C}$, $\mathcal{C}$ rigid tensor category, denote by $\langle X \rangle^{\otimes}$ the smallest rigid tensor full subcategory of $\mathcal{C}$ which contains X.

Def. (neutral Tannakian category)

A *neutral Tannakian category* over a field k is a rigid abelian tensor category $(\mathcal{C}, \otimes)$ such that $\mathrm{End}(1) \cong k$ for which there exists an exact faithful k-linear tensor functor

$$\omega : \mathcal{C} \longrightarrow \mathrm{Vect}_k^{fd}.$$

Such an ω is called a *fibre functor*, and we say that *$\mathcal{C}$ is neutralised by ω*.

Note: For an affine group scheme G over k, $(\mathrm{Rep}_k(G), \otimes)$ is neutral Tannakian and is neutralised by the forgetful functor ω^G.

"If you go into a forest for a stroll and you pick up a category and you ask if it is neutral Tannakian, it is hard."

Theorem (Tannakian reconstruction)

Let $(\mathcal{C}, \otimes)$ be a neutral Tannakian category over k neutralised by $\omega : \mathcal{C} \to \mathrm{Vect}_k^{fd}$.
Then $\mathcal{A}ut^{\otimes}(\omega) : \mathrm{Alg}_k \to \mathrm{Set}$ is representable by an affine group scheme G over k and $(\mathcal{C}, \otimes, \omega) \simeq (\mathrm{Rep}_k(G), \otimes, \omega^G)$ as neutralised Tannakian categories.

Def.: We call $\mathcal{A}ut^{\otimes}(\omega)$ the *Tannakian fundamental group* of $\mathcal{C}$.

Def.: Given any $V \in \mathcal{C}$, $\langle V \rangle^{\otimes}$ is neutral Tannakian and we call its Tannakian fundamental group the *monodromy group* of V.

9.4.2024

Correction: $\langle X \rangle$ ... full subcategory of $\mathcal{C}$ on subquotients of finite direct sums of X

$\langle X \rangle^{\otimes}$ ——— " ——— $X^{\otimes r} \otimes (X^{\vee})^{\otimes s}$, $r, s \in \mathbb{N}$

Proposition / Recollection

Let A be a coalgebra over k, $\omega^A : \mathrm{Comod}^{fd}_A \to \mathrm{Vect}^{fd}_k$ the forgetful functor.

(i) Assume we have a tensor structure $\otimes$ on Comod^{fd}_A for which (ω^A, id) is a tensor functor. Then A is a commutative algebra compatibly with comodule structure and $\otimes$ is identified with the tensor product on comodules.

(ii) If the tensor structure [as in (i)] on Comod^{fd}_A is rigid, then A is a commutative Hopf algebra so that

$$(\mathrm{Comod}^{fd}_A, \otimes) \simeq (\mathrm{Rep}_k(G), \otimes), \qquad G = \mathrm{Spec}\, A.$$

Proof: Lecture 3.

Theorem (reconstruction of coalgebra)

Let $\mathcal{C}$ be a k-linear abelian category, $\omega : \mathcal{C} \to \mathrm{Vect}^{fd}_k$ an exact faithful k-linear functor. Then there exists a coalgebra $A_{\mathcal{C}}$ over k such that

$$(\mathcal{C}, \omega) \simeq (\mathrm{Comod}^{fd}_{A_{\mathcal{C}}}, \omega^{A_{\mathcal{C}}})$$

[where $\omega^{A_{\mathcal{C}}}$ is the forgetful functor, and $\simeq$ means equivalence compatible w/ ω & $\omega^{A_{\mathcal{C}}}$]

Using this theorem on coalgebras, we can already prove the main result:

Proof of Tannakian reconstruction theorem

- By the above theorem, we get a coalgebra $A_{\mathcal{C}}$ such that

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\simeq"] \arrow[dr, "\omega"'] & \arrow[d, phantom, "\cong"] & \mathrm{Comod}^{fd}_A \arrow[dl, "\omega^{A_{\mathcal{C}}}"] \\
& \mathrm{Vect}^{fd}_k &
\end{tikzcd}$$

- By the proposition [top of this page], $A_{\mathcal{C}}$ is a commutative Hopf algebra and, for $G = \mathrm{Spec}\, A_{\mathcal{C}}$,

$$\begin{tikzcd}
(\mathcal{C}, \otimes) \arrow[r, "\simeq"] \arrow[dr, "\omega"'] & (\mathrm{Comod}^{fd}_{A_{\mathcal{C}}}, \otimes) \arrow[d, phantom, "\cong"] \arrow[r, "\simeq"] & (\mathrm{Rep}_k(G), \otimes) \arrow[dl, "\omega^G"] \\
& \mathrm{Vect}^{fd}_k &
\end{tikzcd}$$

- By the Tannaka-Krein theorem, $G = \mathcal{A}ut^{\otimes}(\omega^G) \cong \mathcal{A}ut^{\otimes}(\omega)$. $\square$

We are now left with proving the reconstruction theorem for coalgebras above.

Proof strategy (reconstruction of coalgebra)

Step 1: $X \in \mathcal{C} \rightsquigarrow R \in \mathrm{Alg}^{fd}_k$ s.t. $\langle X \rangle \simeq \mathrm{Mod}^{fg}_{R^{op}}$ [so right modules].

Step 2: $\rightsquigarrow M \in \mathrm{Mod}^{fg}_R$ s.t.

$$\begin{tikzcd} \langle X \rangle \arrow[rr, "\simeq"] \arrow[dr, "\omega"'] & \arrow[d, phantom, "\wr"] & \mathrm{Mod}^{fg}_{R^{op}} \arrow[dl, "- \otimes_R M"] \\ & \mathrm{Vect}^{fd}_k & \end{tikzcd}$$

Step 3: $A_X := A := M^\vee \otimes_R M$ has a coalgebra structure such that each $N \otimes_R M$ is a comodule and

$$\begin{tikzcd} \mathrm{Mod}^{fg}_{R^{op}} \arrow[rr, "- \otimes_R M", "\sim"'] \arrow[dr, "- \otimes_R M"'] & \arrow[d, phantom, "\wr"] & \mathrm{Comod}^{fd}_A \arrow[dl, "\omega^A"] \\ & \mathrm{Vect}^{fd}_k & \end{tikzcd}$$

[commutativity is clear]

Step 4: Express $\mathcal{C}$ as a directed union of $\langle X \rangle$, $X \in \mathcal{C}$, and take the colimit of the coalgebras A_X.

So: Step 1 is about "projective generators and Morita equivalence".

↳ Let $\mathcal{A}$ be an abelian category. ~~Call~~ We say that $\mathcal{A}$ is ~~of~~ finite length if every object of $\mathcal{A}$ has a finite composition series. For $P \in \mathcal{A}$ we ~~a~~ define:

- P projective $:\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{A}}(P,-)$ exact
- P generator $:\Leftrightarrow$ $\mathrm{Hom}_{\mathcal{A}}(P,-)$ faithful.

Note: P projective generator $\Leftrightarrow$ P projective and $\forall A \in \mathcal{A}: \mathrm{Hom}_{\mathcal{A}}(P,A) \neq 0$.

Note, also: If $\mathcal{A}$ is ~~of~~ finite length and $P \in \mathcal{A}$ a projective generator, then for every $A \in \mathcal{A}$ there exists a surjection $P^{\oplus r} \twoheadrightarrow A$ for some $r \in \mathbb{N}$.

Lemma (Morita equivalence)
$\mathcal{A}$ finite length, P projective generator $\Rightarrow$

$$\mathcal{A} \xrightarrow{\simeq} \mathrm{Mod}^{fg}_{\mathrm{End}_{\mathcal{A}}(P)^{op}}$$
$$A \longmapsto \mathrm{Hom}_{\mathcal{A}}(P,A).$$

Proof: · $\mathrm{End}_{\mathcal{A}}(P)$ acts on each $\mathrm{Hom}_{\mathcal{A}}(P,A)$ by precomposition, which gives the module structure.

· Given $P^{\oplus r} \twoheadrightarrow A$, projectivity yields ~~Ho~~ $\mathrm{End}_{\mathcal{A}}(P)^{\oplus r} \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(P,A)$, so the functor $\mathrm{Hom}_{\mathcal{A}}(P,-)$ really goes to $\mathrm{Mod}^{fg}_{\mathrm{End}_{\mathcal{A}}(P)^{op}}$.

· Fullness: start with $A, B \in \mathcal{A}$ and surjections $P^{\oplus r} \twoheadrightarrow A$, $P^{\oplus s} \twoheadrightarrow B$. Suppose given $\phi: \mathrm{Hom}_{\mathcal{A}}(P,A) \to \mathrm{Hom}_{\mathcal{A}}(P,B)$.

↳ We then have the setting

$$\begin{tikzcd} \operatorname{End}_{\mathcal{A}}(P)^{\oplus r} \arrow[r, two heads] \arrow[d, dotted, "\Psi"'] & \operatorname{Hom}_{\mathcal{A}}(P,A) \arrow[d, "\Phi"] \\ \operatorname{End}_{\mathcal{A}}(P)^{\oplus s} \arrow[r, two heads] & \operatorname{Hom}_{\mathcal{A}}(P,B) \end{tikzcd} \quad , \quad \begin{tikzcd} P^{\oplus r} \arrow[r, two heads] \arrow[d, dotted, "\bar\Psi"'] & A \arrow[d, dotted, "\bar\Phi"] \\ P^{\oplus s} \arrow[r, two heads] & B \end{tikzcd}$$

by freeness

Ψ is a matrix of endos, and that can go here.

check that $\bar\Phi$ factors

- Essential surjectivity: ~~For $N \in \mathcal{A}$, let $M := \operatorname{Hom}_{\mathcal{A}}(P,N)$.~~

  Given $M \in \operatorname{Mod}^{fg}_{\operatorname{End}_{\mathcal{A}}(P)^{op}}$, write it (by Noetherianity) as a cokernel

$$\operatorname{End}(P)^{\oplus s} \to \operatorname{End}(P)^{\oplus r} \to M \to 0 .$$

By fullness, this comes from a ~~sequence~~ morphism $P^{\oplus s} \to P^{\oplus r}$, with cokernel $\overset{\text{defined to be}}{N}$, i.e. $P^{\oplus s} \to P^{\oplus r} \to N \to 0$. Now take $\operatorname{Hom}_{\mathcal{A}}(P,-)$ of this exact sequence.

Lemma (Gabber)

Let $\mathcal{A}$ be a k-linear abelian category of finite length such that each $\operatorname{Hom}_{\mathcal{A}}(A,B)$ is a finite-dimensional k-vector space. Then for each $X \in \mathcal{A}$, $\langle X \rangle$ has a projective generator.

Proof idea: $S :=$ finite set of simple constituents of X [i.e. the composition factors].

Fact: $\forall S \in \mathcal{S}$, there is $P_S \twoheadrightarrow S$ with P_S projective. To construct these, need "essential extensions" and induction on length of X.

Take $P := \bigoplus_{S \in \mathcal{S}} P_S$. [see Szamuely, 6.5.5]

→ Altogether this concludes step 1: $X \in \mathcal{C}$, P proj. gen. of $\langle X \rangle$, $R := \operatorname{End}_{\mathcal{C}}(P)$ gives $\langle X \rangle \simeq \operatorname{Mod}^{fg}_{R^{op}}$.

→ Step 2 - "chasing down ω".

Take $M := P$. Claim:

$$\begin{tikzcd} \langle X \rangle \arrow[rr, "\operatorname{Hom}(P,-)", "\simeq"'] \arrow[dr, "\omega"'] & \arrow[d, phantom, "\cong"] & \operatorname{Mod}^{fg}_{R^{op}} \arrow[dl, "-\otimes_R P"] \\ & \operatorname{Vect}^{fd}_k & \end{tikzcd}$$

[$R = \operatorname{End}_{\mathcal{C}}(P)$ as above]

Proof: We have a natural map, for $A \in \langle X \rangle$,

$$\operatorname{Hom}(P,A) \otimes_{\operatorname{End}(P)} \omega(P) \longrightarrow \omega(A).$$

It's an iso if $A = P$, hence if $A = P^{\oplus r}$. For arbitrary A, take $P^{\oplus r} \twoheadrightarrow A$ with kernel K, which yields

$$\begin{tikzcd} \omega(K) \arrow[r] & \omega(P^{\oplus r}) \arrow[r] & \omega(A) \arrow[r] & 0 \\ \operatorname{Hom}(P,K)\otimes\omega(K) \arrow[r] \arrow[u, "2.\text{ also surjective}"'] & \operatorname{Hom}(P,P^{\oplus r})\otimes_R \omega(P) \arrow[r] \arrow[u, "\cong"'] & \operatorname{Hom}(P,A)\otimes_R \omega(A) \arrow[r] \arrow[u, "\substack{1.:\text{ surjective} \\ 3.:\text{ injective (5-lemma)}}"'] & 0 \end{tikzcd}$$

//

<u>Note</u> $-\otimes_R M$ is exact and fully faithful [by assumption on ω].

<u>Step 3</u> ("Barr-Beck reasoning")

$A_X := A := M^\vee \otimes_R M$ [$M^\vee$ the vector space dual]. Let $\delta: k \to M^\vee \otimes_R M$ be the coevaluation.

For every $N \in \mathrm{Mod}^{fg}_{R^{op}}$: $N \otimes_R M \xrightarrow{id_{N \otimes_R M} \otimes \delta} N \otimes_R M \otimes_k M^\vee \otimes_R M$

- For $N = M^\vee$, this makes A a coalgebra (with counit $\varepsilon: M^\vee \otimes_R M \to k$),
- and arbitrary N become comodules.

<u>Claim</u>: This construction defines an equivalence

$$\begin{tikzcd} \mathrm{Mod}^{fg}_{R^{op}} \arrow[rr, "\simeq"] \arrow[dr, "\otimes_R M"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathrm{Comod}^{fd}_A \arrow[dl, "\omega^A \text{ (forgetful)}"] \\ & \mathrm{Vect}^{fd}_k & \end{tikzcd}$$

[only the "$\simeq$" needs proving]

<u>Proof idea</u>: Need to check full faithfulness and essential surjectivity.

- faithfulness clear.
- for essential surjectivity, use exactness of $-\otimes_R M$ and the comodule exact sequence. [Szamuely 6.5.12]

<u>Step 4</u> – "taking the colimit".

Say $\langle Y\rangle \subseteq \langle X\rangle$. So far we've seen

$$\underbrace{\begin{array}{ccccc} \langle X\rangle & \simeq & \mathrm{Mod}^{fg}_{R^{op}} & \simeq & \mathrm{Comod}^{fd}_{A_X} \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| \\ \langle Y\rangle & \simeq & \mathrm{Mod}^{fg}_{(R/I)^{op}} & \simeq & \mathrm{Comod}^{fd}_{A_Y} \end{array}}_{\text{steps 1,2} \qquad\qquad \text{step 3}},$$

and we can ~~unravel~~ track the constructions to see that $A_Y \hookrightarrow A_X$ as coalgebras.

Let $A_{\mathcal{C}} := \operatorname{colim}_{X \in \mathcal{C}} A_X$, a directed union via $\langle X\rangle, \langle Y\rangle \subseteq \langle X \oplus Y\rangle$. We obtain from this a map

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{Comod}^{fd}_{A_{\mathcal{C}}} \\ X &\longmapsto \big(\omega(X) \text{ as comodule via } \\ &\qquad \omega(X) \to \omega(X) \otimes A_X \to \omega(X) \otimes A_{\mathcal{C}}\big). \end{aligned}$$

To check that it is fully faithful, it suffices to work in $\langle X \oplus Y\rangle$ for $X, Y \in \mathcal{C}$, where we have already seen it. For essential surjectivity, note that any $V \in \mathrm{Comod}^{fd}_{A_{\mathcal{C}}}$ is a comodule for some finite-dimensional $A_V \subseteq A_{\mathcal{C}}$, and thus A_V must lie in some A_X, $X \in \mathcal{C}$.

3. Some examples

Example (graded vector spaces)

$\mathcal{C} := \{\mathbb{Z}\text{-graded f.d. v.s.} / k\}$, objects $\bigoplus_{i \in \mathbb{Z}} V^i \cdot t^i$ (t a formal variable)

This is neutral Tannakian via the forgetful functor. It is generated by the [or a] ~~f.d. v.s.~~ 1D v.s. in degree 1, i.e. $\mathcal{C} = \langle k \cdot t^1 \rangle^{\otimes}$.

$\rightsquigarrow \underline{Aut}^{\otimes}(\omega)(R) = \underline{Aut}_{Mod_R}(R) = R^\times = \mathbb{G}_m(R)$, so $(\mathcal{C}, \otimes) \simeq (Rep_k(\mathbb{G}_m), \otimes)$.

Endo~~Auto~~morphisms of $(\mathcal{C}, \otimes)$ are given by mapping $k \cdot t^1$ to any other invertible object. This ~~in particular~~ gives an integer $n \in \mathbb{Z}$ – the degree of that invertible object. Indeed, $\underset{\text{End}}{\cancel{\text{Aut}}}(\mathbb{G}_m) \cong \mathbb{Z}$ via $t \mapsto t^n$.

Example (diagonalisable groups)

M a f.g. abelian group, so $M \cong \mathbb{Z}^{\oplus r} \oplus \bigoplus_i (\mathbb{Z}/p_i^{m_i}\mathbb{Z})^{\oplus a_i}$.

Let $\mathcal{C} := \{M\text{-graded f.d. v.s.} / k\}$, neutral Tannakian.

$\rightsquigarrow (\mathcal{C}, \otimes) \simeq: (Rep_k(D(M)), \otimes)$

for some affine group scheme $D(M)$ over k. Explicitly, $A := k[M]$ (group algebra) gives $D(M) = \operatorname{Spec} A$, and so

$$D(M) \cong \mathbb{G}_m^{\times r} \times \prod_i \mu_{p_i^{m_i}}^{\times a_i}$$

Example (real Hodge structures)

$Hod_{\mathbb{R}} := \{\text{real Hodge structures}\}$, that is:

- objects: $(V, V^{p,q})$ where $V \in Vect_{\mathbb{R}}^{fd}$ & $V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q} V^{p,q}$ such that $\overline{V^{p,q}} = V^{q,p}$.

- fibre functor $\omega : (V, V^{p,q}) \mapsto V$.

This is ~~the~~ a neutralised Tannakian category. Let $\mathbb{S} := Res_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_m$ [Weil restriction], i.e. on the functor of points it ~~does~~ sends $B \in Alg_{\mathbb{R}}$ to $\mathbb{S}(B) := \mathbb{G}_m(B \otimes_{\mathbb{R}} \mathbb{C})$.

[This is also called the Deligne torus]. Then $\mathbb{S}$ is an affine group scheme over $\mathbb{R}$; its base change to $\mathbb{C}$ is $\mathbb{G}_m^{\times 2}$. Indeed, we have

$$(Hod_{\mathbb{R}}, \otimes) \simeq (Rep_{\mathbb{R}}(\mathbb{S}), \otimes)$$

Example (groups of multiplicative type)

Let G be an affine group scheme over a field k such that $G_{k^{sep}}$ is diagonalisable.

Then G is the Tannakian fundamental group of

⌜Or put k' a finite extension in place of k^{sep}!⌟

$$\mathcal{C} = \{ V \in \mathrm{Vect}_k^{fd} \text{ with } M\text{-grading on } V \otimes_k k^{sep} \text{ compatible with the action of } \mathrm{Gal}(k^{sep}/k) \}$$

where M is s.t. $G_{k^{sep}} = D(M)$.

Example (topological groups)

K a topological group, $\mathrm{Rep}_{\mathbb{R}}^{top}(K) := \{\text{f.d. real cts. reps. of } K\}$ is neutral Tannakian.

$\rightsquigarrow$ get as Tannakian group a real affine group scheme $K_{\mathbb{R}}^{alg}$, called the <u>real algebraic envelope</u> such that $(\mathrm{Rep}_{\mathbb{R}}^{top}(K), \otimes) \simeq (\mathrm{Rep}_{\mathbb{R}}(K_{\mathbb{R}}^{alg}), \otimes)$.

Similarly for complex representations & $K_{\mathbb{C}}^{alg}$ (now over $\mathbb{C}$).

Example (abstract groups)

H abstract group, $\mathcal{C} = \{\text{representations of } H \text{ on f.d. v.s.}\}$, neutral Tannakian.

$\rightsquigarrow H^{alg}$ <u>algebraic envelope</u> of H s.t. $(\mathcal{C}, \otimes) \simeq (\mathrm{Rep}_k(H^{alg}), \otimes)$. If H is finite, then H^{alg} is the associated discrete affine group scheme. In general, assuming $k = \bar{k}$, we can associate to a rep. $\rho: H \to GL(V)$ the Tannakian monodromy group of V, and it is given by the Zariski closure of $\mathrm{im}(\rho) \leq GL(V)$.

Example (local systems)

X a connected, locally simply connected topological space, $x_0 \in X$ base point, k field.

$\mathrm{Loc}_k(X) := \{\text{local systems on } X \text{ with coefficients in } k\}$ is Tannakian,

∟ objects: loc. cst. sheaves of f.d. k-v.s. on X

neutralised by $\omega: \mathcal{L} \mapsto \mathcal{L}_{x_0}$. Its Tannakian fundamental group is the algebraic envelope of $\pi_1(X, x_0)$. Moreover, given $\mathcal{L} \in \mathrm{Loc}_k(X)$, the Tannakian fundamental group of $\langle \mathcal{L} \rangle^{\otimes}$ is the Zariski closure of the image of the monodromy representation $\rho_{\mathcal{L}, x_0}: \pi_1(X, x_0) \to GL(\mathcal{L}_{x_0})$.

Example (Artin motives)

k field (of char. 0)

X smooth projective /k $\rightsquigarrow$ $h(X) \in M_k$ ····· $\mathbb{Q}$-linear Tannakian category of motives over k, Tannakian

motive of X

Let's specialise to $M_k^0 := \{$motives of 0-dim. varieties over $k\}$

$\rightsquigarrow$ finite field extensions (automatically separable by char = 0)

with fibre functor $\omega : h(X) \longmapsto \bigoplus_{x \in X(\bar{k})} \mathbb{Q}$.

Then $M_k^0 \simeq \{$continuous reps of $Gal(\bar{k}/k)$ on f.d. $\mathbb{Q}$-v.s.$\}$ and the Tannakian fundamental group recovers this representation theory.